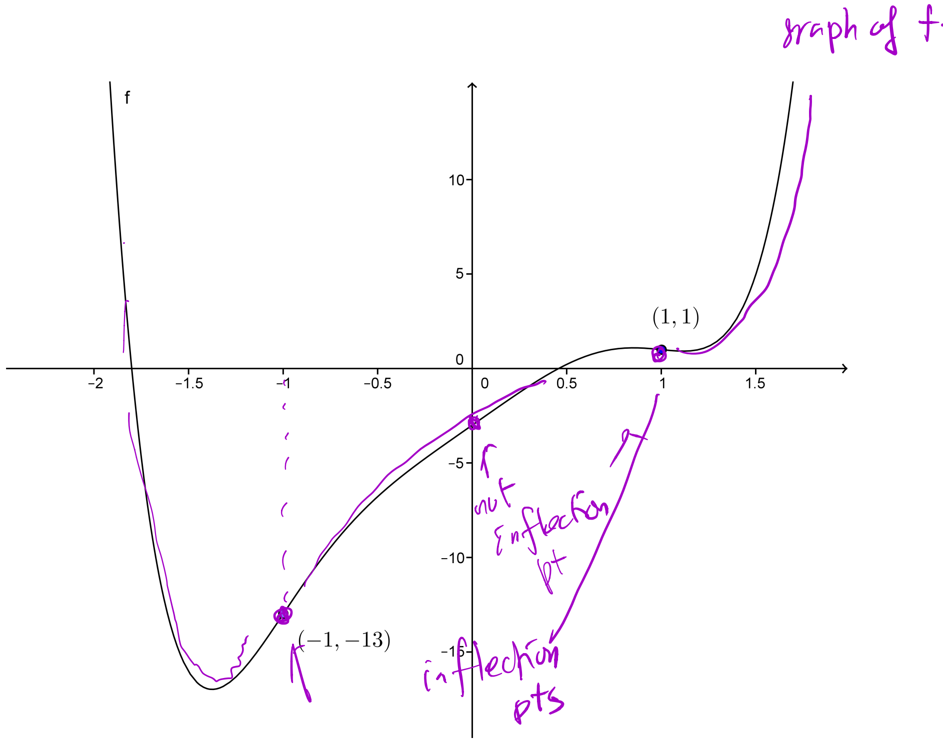
points.



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Remark.

- c is a critical point $\iff$ $f'(c) = 0$ or $f'(c)$ does not exist
  c is a critical point $\left\{\begin{matrix}\Longleftarrow \\ \not\Longrightarrow\end{matrix}\right\}$ f' changes sign at c
- $(c, f(c))$ is an inflection point $\iff$ f'' changes sign at c
  $(c, f(c))$ is an inflection point $\left\{\begin{matrix}\Longrightarrow \\ \not\Longleftarrow\end{matrix}\right\}$ $f''(c) = 0$ or $f''(c)$ does not exist

Theorem 7.1.1 (The Second Derivative Test: Relative Extrema).

Suppose $f'(a) = 0$*!*

1. *If* $f''(a) < 0$*, then* f *has a relative maximum at* a*.*
2. *If* $f''(a) > 0$*, then* f *has a relative minimum at* a*.*
3. *If* $f''(x) = 0$*, we have no conclusion.*

	relative min	relative min	relative max	relative max
derivative	$f'(a) = 0$	$f'(a)$ does not exist	$f'(a)$ does not exist	$f'(a) = 0$
1st test:	$-\quad+$	$-\quad+$	$+\quad-$	$+\quad-$
2nd test:	$f''(a) > 0$	Not Applicable	Not Applicable	$f''(a) < 0$
derivative				

Example 7.1.3.

$$f(x) = \frac{1}{30}x^6 - \frac{1}{12}x^4.$$

Use the first and second derivative test to study the relative extrema.

Solution. $\frac{6}{30}x^5 - \frac{4}{12}x^3$ $\quad \frac{x^3}{5}\left(x^2 - \frac{5}{3}\right)$

$$f'(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3 = \frac{1}{5}x^3\left(x + \sqrt{\frac{5}{3}}\right)\left(x - \sqrt{\frac{5}{3}}\right) = 0 \quad \Rightarrow \quad x = -\sqrt{\frac{5}{3}}, 0, \sqrt{\frac{5}{3}}$$

critical points of f

$$f''(x) = x^2(x+1)(x-1). = \frac{1}{1}x^4 - \frac{1}{3}x^2 = x^2(x^2-1)$$

crit pt $\qquad$ crit $\qquad$ crit

x	$(-\infty, -\sqrt{\frac{5}{3}})$	$-\sqrt{\frac{5}{3}}$	$(-\sqrt{\frac{5}{3}}, 0)$	0	$(0, \sqrt{\frac{5}{3}})$	$\sqrt{\frac{5}{3}}$	$(\sqrt{\frac{5}{3}}, +\infty)$
$f'(x)$	$(-)(-)(-)$ $-$	0	$(-)(+)(-)$ $+$	0	$-$	0	$+$
$f''(x)$		$f'' > 0$		$f'' = 0$		$f'' > 0$	
1st test:		relative min		relative max		relative min	
2nd test:		relative min		inconclusive		relative min	

$f''(-\sqrt{\frac{5}{3}}) = (+)(-)(-) = (+)$ $\qquad$ $f''(0) = 0$ $\qquad$ $f''(\sqrt{\frac{5}{3}}) = (+)(+)(+) = (+)$ ■

$\sqrt{\frac{5}{3}} > 1$

Exercise 7.1.1. Apply the first and second derivative test for $f(x) = x^3 - 3x$.

$f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1)$.

$\rightsquigarrow$ $f'(x) = 0$ when $x = \pm 1$ $\leftarrow$ critical points of f candidates for rel extrema

$f''(x) = 6x$ $\qquad$ $f''(1) = 6 > 0$ $\rightsquigarrow$ $x=1$ is a local min.

$f''(-1) = -6 < 0$ $\rightsquigarrow$ $x=-1$ is a local max.

7.2 Curve sketching

undefined when $=1$

$\lim_{x\to 1^+} f(x) = 1 + \frac{1}{0^+} = +\infty$

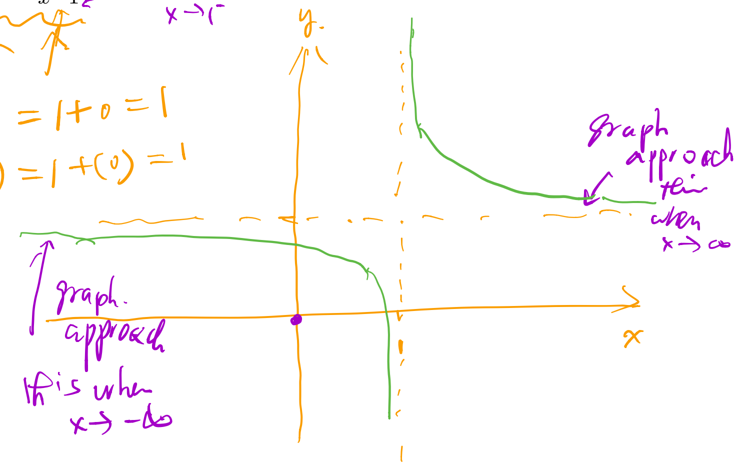
$\lim_{x\to 1^-} f(x) = 1 + \frac{1}{0^-} = -\infty$

Example 7.2.1. Sketch the graph of $y = f(x) = 1 + \frac{1}{x-1}$.

Solution.

$\lim_{x\to\infty} f(x) = 1 + 0 = 1$

$\lim_{x\to -\infty} f(x) = 1 + (0) = 1$



Step 1. Analyze $f(x)$.

1. domain: $\{x \in \mathbb{R} \,|\, x \neq 1\}$
2. x, y intercepts:
   Let $x = 0$, then $y = 0$;
   Let $y = 0$, then $x = 0$.
   $\Rightarrow$ only one intercept: $(0, 0)$

   $y = 1 + \frac{1}{-1} = 0$

   $-1 = \frac{1}{x-1}$
3. vertical and horizontal asymptotes:

$$\lim_{x\to 1^+} f(x) = +\infty, \lim_{x\to 1^-} f(x) = -\infty \quad \Rightarrow \quad \text{vertical asymptote: } x = 1$$
$$\lim_{x\to +\infty} f(x) = 1, \lim_{x\to -\infty} f(x) = 1 \quad \Rightarrow \quad \text{horizontal asymptote: } y = 1.$$

Step 2. Analyze $f'(x)$.

$$f'(x) = -\frac{1}{(x-1)^2}, x \neq 1.$$

f is strictly decreasing in its natural domain

wrap going down

1. interval where f is strictly increasing: none $\quad (f'(x) < 0$ in the domain)
   interval where f is strictly decreasing: $(-\infty, 1)$, $(1, +\infty)$
2. critical points of f: ~~none~~ $\quad (x = 1$ is not in the domain)
3. relative extrema of f: none

Step 3. Analyze $f''(x)$.

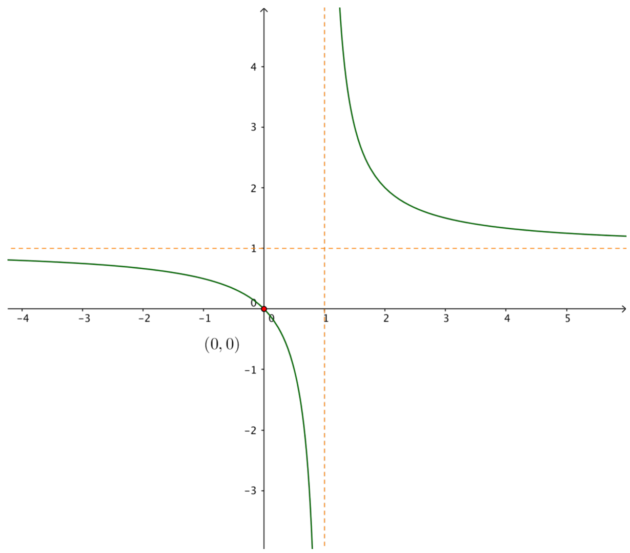
$-1\left(-2 \frac{1}{(x-1)^3}\right)$

$$f''(x) = \frac{2}{(x-1)^3}, x \neq 1.$$

1. interval where f is strictly convex: $(1, +\infty) \quad (f'' > 0)$
   interval where f is strictly concave: $(-\infty, 1) \quad (f'' < 0)$
2. inflection points on the graph: none $\quad (x = 1$ is not in the domain)

$f'' \neq 0$ on its domain

Step 4. Sketch.



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Definition 7.2.1 (Asymptotes)**.**

the line $x = c$ is a vertical asymptote of the graph of $f(x)$

$$\text{if } \lim_{x\to c^-} f(x) \text{ or } \lim_{x\to c^+} f(x) \text{ is } +\infty \text{ or } -\infty;$$

the line $y = b$ is called a horizontal asymptote of the graph of $f(x)$

$$\text{if } \lim_{x\to -\infty} f(x) \text{ or } \lim_{x\to +\infty} f(x) \text{ is } b.$$

Note: It may happen that both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, but they are not the same.

A General Procedure for Sketching the Graph of $f(x)$

the graph does not intersect vertical lines $x=a$ when a is not in the domain

Step 1. Analyze $f(x)$:

(1) domain, (2) x, y intercepts, (3) vertical / horizontal asymptotes of the graph.

Step 2. Analyze $f'(x)$:

(1) intervals where f is increasing / decreasing, (2) critical points of f (3) relative extrema of f

Step 3. Analyze $f''(x)$:

(1) intervals of where f is convex/concave, (2) inflection points on the graph

$f'' < 0 / > 0$

$f'' = 0$ and changes sign nearby

Step 4. Sketch:

First label all asymptotes, intercepts, critical points, inflection points, then sketch the graph.

Example 7.2.2. Sketch the graph of

$$f(x) = \frac{x}{(x+1)^2}.$$

Solution.

Step 1. Analyze $f(x)$.

1. domain: $\{x \in \mathbb{R} \mid x \neq -1\}$
2. x, y intercepts:
   Let $x = 0$, then $y = 0$;
   Let $y = 0$, then $x = 0$.
   $\Rightarrow$ only one intercept: $(0, 0)$
3. vertical and horizontal asymptotes:

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = -\infty \quad \Rightarrow \quad \text{vertical asymptote: } x = -1$$
$$\lim_{x\to +\infty} f(x) = \lim_{x\to -\infty} f(x) = 0 \quad \Rightarrow \quad \text{horizontal asymptote: } y = 0.$$

Step 2. Analyze $f'(x)$.

$$f'(x) = \frac{1-x}{(x+1)^3} = 0 \quad \Rightarrow \quad x = 1.$$

x	$(-\infty, -1)$	$(-1, 1)$	1	$(1, +\infty)$
$f'(x)$	$-$	$+$	0	$-$
$f(x)$	$\downarrow$	$\uparrow$	max: 1	$\downarrow$

only one critical point: 1 (with corresponding critical value $\frac{1}{4}$), at which a relative maximum occurs. ($x = -1$ is not in the domain.)

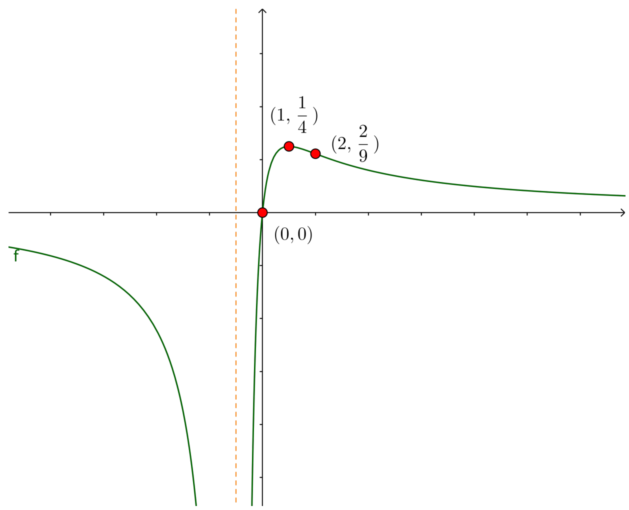
Step 3. Analyze $f''(x)$.

$$f''(x) = \frac{2(x-2)}{(x+1)^4} = 0 \quad \Rightarrow \quad x = 2.$$

x	$(-\infty,-1)$	$(-1,2)$	2	$(2,+\infty)$
$f''(x)$	$-$	$-$	0	$+$
graph of $f(x)$	$\frown$	$\frown$	inflection point	$\smile$

inflection point: $(2,\frac{2}{9})$

Step 4. Sketch.



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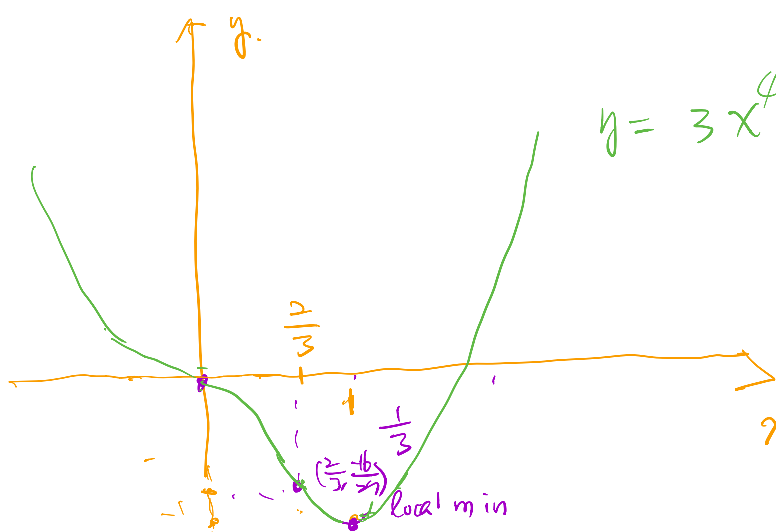
Exercise 7.2.1. Sketch the graph of $3x^4-4x^3$.

$f(x) = 3x^4 - 4x^3 = x^3(3x-4)$ well defined on $\mathbb{R}$



$y = 3x^4 - 4x^3$

- the graph $y = x^3(3x-4)$ intersects the x-axis when $y = 0 = x^3(3x-4)$

  $\rightsquigarrow$ $x = 0$ or $\frac{4}{3}$

  intersects the y-axis when $x = 0$

$\rightsquigarrow \; y = x^3(3x-4) = 0$

- horizontal asymptotes

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty}\left(x^3(3x-4)\right) = +\infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\underset{\substack{\uparrow\\-\infty}}{x^3}\;\underset{\substack{\uparrow\\-\infty}}{(3x-4)}\right) = \infty$$

no horizontal asymptotes.

vertical asymptotes :

are there finite a such that $\lim_{x\to a^+} f(x) = \pm\infty$

or $\lim_{x\to a^-} f(x) = \pm\infty$

No! so there are no vertical asymptotes

- $f'(x) = (3x^4 - 4x^3)' = \underline{12x^3 - 12x^2} = 12x^2(x-1)$

$f' = 0$ when $x = 0, 1$. (critical points potential local extrema)

$f' < 0$ $\quad$ $f' < 0$ $\quad$ $f' > 0$

x: 0, 1

f decreasing $\quad$ decreasing $\quad$ increasing

local min.

$f(1) = 3\cdot 1 - 4\cdot 1 = -1$

- $f''(x) = 36x^2 - 24x = \underline{12x(3x-2)}$

$f'' = 0$ when $x = 0$ or $\frac{2}{3}$ (potential inflection points)

inflection pts

$f(0) = 0$ $\quad$ $f\left(\frac{2}{3}\right) = 3\cdot\frac{2^4}{3^4} - 4\cdot\frac{2^3}{3^3} = \frac{2^4 - 2^5}{3^3}$

$= \frac{-16}{27}$

$12x < 0$ $\quad$ $12x > 0$

$f'' > 0$ $\quad$ 0 $\quad$ $f'' < 0$ $\quad$ $\frac{2}{3}$ $\quad$ $f'' > 0$ $\quad$ x

$3x - 2 < 0$ $\quad$ $3x - 2 > 0$

f convex $\quad$ f concave $\quad$ f convex